MATH 2050 - Field and order properties of $\mathbb{R}$

(Reference: Bartle §2.1)

Grand Thm: $\mathbb{R}$ is a complete ordered field.

complete — analysis (topology)

ordered — inequalities

field — algebra

Field Properties

Def$^{\underline{n}}$/ Thm: $(\mathbb{R}, +, \cdot)$ is a **field**, i.e.

$\exists$ two operations $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ s.t.

the following properties hold:

$+$:

(A1) $a + b = b + a \quad \forall a, b \in \mathbb{R}$

(A2) $(a+b)+c = a+(b+c) \quad \forall a, b, c \in \mathbb{R}$

(A3) $\exists\, 0 \in \mathbb{R}$ s.t. $0 + a = a = a + 0 \quad \forall a \in \mathbb{R}$

(A4) $\forall a \in \mathbb{R}$, $\exists -a \in \mathbb{R}$ s.t. $a + (-a) = 0 = (-a) + a$.

$\cdot$:

(M1) $a \cdot b = b \cdot a \quad \forall a, b \in \mathbb{R}$

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in \mathbb{R}$

(M3) $\exists\, 1 \in \mathbb{R}$ s.t. $1 \neq 0$ and $1 \cdot a = a = a \cdot 1 \quad \forall a \in \mathbb{R}$

(M4) $\forall a \in \mathbb{R}$, $a \neq 0$, $\exists \frac{1}{a} \in \mathbb{R}$ s.t. $\frac{1}{a} \cdot a = 1 = a \cdot \frac{1}{a} \quad \forall a \in \mathbb{R}$

$+$, $\cdot$:

(D) $a \cdot (b+c) = a \cdot b + a \cdot c \quad \forall a, b, c \in \mathbb{R}$

$(b+c) \cdot a = b \cdot a + c \cdot a \quad \forall a, b, c \in \mathbb{R}$

Note: The remaining algebraic properties can be deduced from the field properties above.

Define: $a-b := a+(-b)$, and if $b\neq 0$, $\dfrac{a}{b} := a\cdot\left(\dfrac{1}{b}\right)$

Notation: $a^n := \underbrace{a\cdot a\cdot\cdots a}_{n \text{ times}}$; $a^0 := 1$; $a^{-1} := \dfrac{1}{a}$

$n\in\mathbb{N}$

Prop: "Cancellation Laws"

(1) $a+c=b+c \Rightarrow a=b$

(2) $ac=bc$, $c\neq 0$ $\Rightarrow a=b$

Proof: (1)

$$
\begin{aligned}
a &= a+0 && \text{(by (A3))}\\
&= a+(c+(-c)) && \text{(by (A4))}\\
&= (a+c)+(-c) && \text{(by (A2))}\\
&= (b+c)+(-c) && \text{(by assumption)}\\
&= b+(c+(-c)) && \text{(by (A2))}\\
&= b+0 && \text{(by (A4))}\\
&= b && \text{(by (A3))}
\end{aligned}
$$

$a+c=b+c$

$\Downarrow$

$a+(c+(-c))=b+(c+(-c))$

$\Downarrow$

$a+0=b+0$

$\Downarrow a=b$

(2): Exercise. $\square$

Cor: The zero element 0 in (A3) is unique.

Proof: Suppose there are two zero elements 0, $0'$. Then

$$0 \underset{0'\ (A3)}{=} 0+0' \underset{0\ (A3)}{=} 0' \qquad \text{i.e. } 0=0'$$

$\square$

Exercise: 1 in (M3) is unique.

Exercise: The additive and multiplicative inverse in (A4) and (M4) are unique.

Prop: (1) $0 \cdot a = 0 \quad \forall a \in \mathbb{R}$

(2) $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$ (or both)

(3) $(-1) \cdot a = -a \quad \forall a \in \mathbb{R}$

Proof: (1) Consider

$$\cancel{0 \cdot a} + 0 \cdot a \overset{(D)}{=} (0+0) \cdot a \overset{(A3)}{=} 0 \cdot a \overset{(A3)}{=} \cancel{0 \cdot a} + 0$$

then by cancellation law (1), we have $0 \cdot a = 0$.

(2) Suppose $a \cdot b = 0$.

Case i: $a = 0 \quad \Rightarrow$ Done.

Case ii: $a \neq 0 \quad$ [Want: Prove $b = 0$.]

Since $a \neq 0$, the inverse $\frac{1}{a} \in \mathbb{R}$ exists.

$$\cancel{a} \cdot b \underset{\text{by assumption}}{=} 0 \underset{\text{by (1)}}{=} \cancel{a} \cdot 0$$

By cancellation law (2), we have $b = 0$.

(3) Want to show: $a + (-1) \cdot a = 0$

Then, result follows from uniqueness of additive inverse "$-a$".

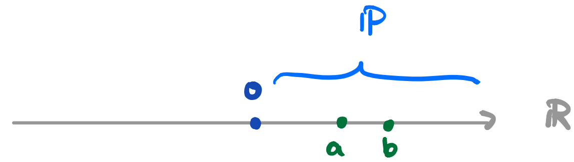
$$\begin{aligned} a + (-1) \cdot a &\overset{(M3)}{=} 1 \cdot a + (-1) \cdot a \\ &\overset{(D)}{=} (1 + (-1)) \cdot a \\ &\overset{(A4)}{=} 0 \cdot a \overset{\text{by (1)}}{=} 0 \end{aligned}$$

■

Remark: Other e.g. of fields $\mathbb{Q}, \mathbb{C}, \mathbb{Z}_p, \left\{ \frac{\text{polynomials}}{\text{polynomials}} \right\}$ ......

Goal: $\mathbb{R}$ is a complete ordered field. ✓

Ordering $\rightsquigarrow$ $\mathbb{R}$ as a real line



Def^{n} / Thm: $\exists\ \phi \neq \mathbb{P} := \{\text{"positive" real numbers}\} \subseteq \mathbb{R}$ s.t.

(O1) : $a, b \in \mathbb{P} \Rightarrow a+b,\ ab \in \mathbb{P}$

(O2) : Trichotomy : $\forall\, a \in \mathbb{R}$, one and only one of the following holds:

$$a \in \mathbb{P} \quad \text{or} \quad a = 0 \quad \text{or} \quad -a \in \mathbb{P}$$

Notation: $a > 0$ if $a \in \mathbb{P}$; $a \geq 0$ if $a \in \mathbb{P} \cup \{0\}$

$a < 0$ if $-a \in \mathbb{P}$; $a \leq 0$ if $-a \in \mathbb{P} \cup \{0\}$

Define: $a > b$ if $a - b \in \mathbb{P}$

$a \geq b$ if $a - b \in \mathbb{P} \cup \{0\}$

Prop: (Rules of inequalities) Let $a, b, c \in \mathbb{R}$.

(a) $a > b$ and $b > c \Rightarrow a > c$

(b) $a > b \Rightarrow a + c > b + c$.

(c) $a > b \Rightarrow \begin{cases} ac > bc & \text{if } c > 0 \\ ac < bc & \text{if } c < 0 \end{cases}$

Proof: (a) By def^{n}, $a > b \iff a - b \in \mathbb{P}$

also $b > c \iff b - c \in \mathbb{P}$

By (O1), $a - c \underset{(A2),(A3),(A4)}{=} \underset{\mathbb{P}}{(a-b)} + \underset{\mathbb{P}}{(b-c)} \in \mathbb{P} \Rightarrow a > c$.

(b) Exercise.

(c) By def^{n}, $a > b \iff a - b \in \mathbb{P}$.

Given $c > 0$, ie. $c \in \mathbb{P}$, then by (O1)

$$ac - bc \overset{(D)}{=} \underset{\in\mathbb{P}}{(a-b)} \cdot \underset{\in\mathbb{P}}{c} \in \mathbb{P} \implies ac > bc.$$

Exercise for the case $c < 0$. ▫

Thm 1: $\mathbb{P}$ contains all natural numbers, ie $\mathbb{N} \subset \mathbb{P}$

Lemma: $a^2 \geq 0 \quad \forall a \in \mathbb{R}$.

Proof: By (O2), there are 3 possible cases:

Case 1: $a \in \mathbb{P}$

$$a^2 = \underset{\in\mathbb{P}}{a} \cdot \underset{\in\mathbb{P}}{a} \overset{(O1)}{\in} \mathbb{P} \qquad \text{so } a^2 \geq 0.$$

Case 2: $a = 0$

$$a^2 = 0 \cdot 0 = 0 \qquad \text{so } a^2 \geq 0.$$

Case 3: $-a \in \mathbb{P}$

$$a^2 \overset{\text{Ex.}}{=} (-a)^2 = \underset{\in\mathbb{P}}{(-a)} \cdot \underset{\in\mathbb{P}}{(-a)} \overset{(O1)}{\in} \mathbb{P} \qquad \text{so } a^2 \geq 0.$$

▫

Proof of Thm 1: Use M.I. to show $n \in \mathbb{P} \quad \forall n \in \mathbb{N}$.

$n = 1$: $\quad 1 = 1 \cdot 1 = 1^2 \overset{\text{Lemma}}{\geq} 0$ and $1 \neq 0$ (by (M3)).

So, $1 \in \mathbb{P}$

Assume $n = k$ is true, ie. $k \in \mathbb{P}$.

Then $\underset{\in\mathbb{P}}{k} + \underset{\in\mathbb{P}}{1} \in \mathbb{P}$ by (O1), so $n = k+1$ is true. ▫

Thm 2: $0 \le a < \varepsilon \quad \forall \varepsilon > 0 \implies a = 0$.

(i.e. there is no "smallest" positive real number.)

Proof: By Contradiction. Suppose $a \neq 0$, then $a > 0$.

Note that $\frac{1}{2} > 0$ [why? If not, then $-\frac{1}{2} \overset{(O2)}{>} 0$

$\implies \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) = -1 \overset{(O1)}{>} 0$]

False $\because 1 > 0$

By (O1), $\underset{\mathbb{P}}{\frac{1}{2}} \cdot \underset{\mathbb{P}}{a} \in \mathbb{P}$, i.e. $\frac{1}{2} a > 0$.

Choose $\varepsilon = \frac{1}{2} a > 0$, by assumption. $a < \frac{1}{2} a$ False. (Ex: why?) $\square$

Prop: (1) $ab > 0 \implies$ either $a > 0$ and $b > 0$
or $a < 0$ and $b < 0$.

(2) $ab < 0 \implies$ either $a > 0$ and $b < 0$
or $a < 0$ and $b > 0$.

Pf: Exercise.